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Fusion of RSOS Models as a Coset Construction

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Dedicated to the memory of Sasha Belov

Using the vertex operator approach we show that fusion of the RSOS models can be considered as a kind of coset construction which is very similar to the coset construction of minimal models in conformal field theory. We reproduce the excitation spectrum and S-matrix of the fusion RSOS models in the regime III and show that their correlation functions and form factors can be expressed in terms of those of the ordinary (ABF) RSOS models.

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1. Introduction

The vertex operator (Kyoto) approach to solvable models of statistical mechanics[1,2] makes it possible to express correlation functions and form factors of local quantities in terms of correlation functions of such highly nonlocal objects as vertex operators. These new correlation functions are solutions of the difference Knizhnik–Zamolodchikov equation.[3,4] Recently a great progress has been made by use of the bosonic representation of vertex operators.[5–7] In particular, the bosonic representation for the ordinary (Andrews–Baxter–Forrester, ABF) RSOS models[8] reveals the structure which is very similar to that of the conformal minimal models.[7] This bosonic representation is well investigated and allows one, in principle, to find all correlators and form factors. On the other hand, the l-fusion $(K+1)$-state RSOS model[9] is known to be described by the (q,x)-deformation ($q=x^{K+2}$) of the coset construction[10]

$$N_{kl} \sim \frac{SU(2)_k\times SU(2)_l}{SU(2)_{k+l}}, \qquad k=K-l. \tag{1.1}$$

It is known in conformal field theory that such coset models can be also considered as coset constructions of minimal models[11,12]

$$\frac{SU(2)_k\times SU(2)_l}{SU(2)_{k+l}} \sim \frac{M_k M_{k+1}\ldots M_{k+l-1}}{M_1\ldots M_{l-1}}, \tag{1.2}$$

where $M_n\sim N_{n1}$ is a minimal conformal model with the central charge $c=1-6/(n+2)(n+3)$. This representation allows one to express correlation functions in the coset models in terms of the correlation functions in the minimal models.

In this letter a similar construction is proposed to express correlation functions of the fusion RSOS models in terms of those of the ordinary RSOS models.

2. Ordinary RSOS Models

The ordinary (ABF) RSOS models in the regime III are identified in the vertex operator approach with an elliptic deformation of the holomorphic sector of the minimal models.

Recall that the primary fields of the minimal model M_k are given by the vertex operators[13]

$$\begin{gathered}\phi^{(k)}_{pq}(z)^{m'n'}_{m\ n} : \mathcal{H}_{mn} \to \mathcal{H}_{m'n'},\\ p, m, m' = 1, 2, \ldots, k+1; \quad q, n, n' = 1, 2, \ldots, k+2;\\ |m-p|+1 \le m' \le \min(m+p-1, k+3-m-p),\\ |n-q|+1 \le n' \le \min(n+q-1, k+4-n-q),\end{gathered} \tag{2.1}$$

where $\mathcal{H}_{mn}$ is the state space generated by the Virasoro algebra from the highest weight vector $\phi_{mn}(0)^{m\,n}_{1\,1}|\mathrm{vac}\rangle$.

The fields $\phi_{12}(z)$ and $\phi_{21}(z)$ are of particular interest because they generate the whole set of the vertex operators in fusion. In the deformed case they have a simple meaning. The operator $\phi_{12}(z)^{n'}_{n} \equiv \oplus_m \phi_{12}(z)^{mn'}_{mn}$ (type I vertex operator) describes the half transfer matrix of the ordinary RSOS model.[2,14] The operator $\phi_{21}(z)^{m'}_{m} \equiv \oplus_n \phi_{21}(z)^{m'n}_{m\ n}$ (type II vertex operator) represents the wave function of an elementary excitation.[2,7] Their commutation relations are given by[7]

$$\begin{aligned}\phi_{12}(z_1)^{n'}_{s}\phi_{12}(z_2)^{s}_{n} &= \sum_{s'} W^-_k\left[\begin{matrix} n' & s'\\ s & n\end{matrix}\,\middle|\, \frac{z_1}{z_2}\right]\phi_{12}(z_2)^{n'}_{s'}\phi_{12}(z_1)^{s'}_{n},\\ \phi_{21}(z_1)^{m'}_{r}\phi_{21}(z_2)^{r}_{m} &= \sum_{r'} W^+_k\left[\begin{matrix} m' & r'\\ r & m\end{matrix}\,\middle|\, \frac{z_1}{z_2}\right]\phi_{21}(z_2)^{m'}_{r'}\phi_{21}(z_1)^{r'}_{m},\\ \phi_{12}(z_1)^{m'n'}_{m'n}\phi_{21}(z_2)^{m'n}_{m\ n} &= \tau\left(\frac{z_1}{z_2}\right)\phi_{21}(z_2)^{m'n'}_{m\ n'}\phi_{12}(z_1)^{mn'}_{mn}.\end{aligned} \tag{2.2}$$

Here W^-_k is the weight matrix of the RSOS model, $\tau(z)$ is the elementary excitation spectrum, and W^+_k is the S-matrix of two elementary excitation. An N-particle excitation of the transfer matrix $T^{(k)}(z)$ is described by a chain of integers $(m_0, m_1, m_2, \ldots, m_{N-1}, m_N)$, $m_{i+1} = m_i \pm 1$, $m_i = 1, 2, \ldots, k+1$, and a chain of complex numbers (spectral parameters) $(z_1, z_2, \ldots, z_N)$, $|z_i| = 1$. It is given by an operator $\phi_{21}(z_N)^{1}_{m_{N-1}} \ldots \phi_{21}(z_2)^{m_2}_{m_1}\phi_{21}(z)^{m_1}_{1}$ in the sense of Refs. 10, 2. The respective eigenvalue of the transfer matrix is $\tau(z/z_1)\tau(z/z_2)\ldots\tau(z/z_N)$. Quantities m_0 and m_N define boundary conditions and can be treated as topological numbers of the solution. The boundary number $m_0 = m$ ($m_N = m$) means that we consider configurations stabilizing at the positive (negative) infinity to the sequence $\ldots$, m, $m+1$, m, $m+1$, $\ldots$

The coefficients τ, W^-_k, and W^+_k are expressed explicitly in terms of the functions

$$\begin{gathered}\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty,\\ (z; p_1, \ldots, p_N)_\infty = \prod_{n_1,\ldots,n_N=0}^{\infty}(1 - z p_1^{n_1}\ldots p_N^{n_N})\end{gathered}$$

as follows:[7]

$$\tau(z) = z^{-1/2}\Theta_{x^4}(xz)/\Theta_{x^4}(x^3 z) = f(z^{-1})/f(z), \tag{2.3}$$

$$\begin{aligned}W^-_k\left[\begin{matrix} n_4 & n_3\\ n_1 & n_2\end{matrix}\,\middle|\, z\right] &= z^{\frac{1}{2}\frac{k+2}{k+3}}\frac{g^-_k(z^{-1})}{g^-_k(z)}\hat{W}\left[\begin{matrix} n_4 & n_3\\ n_1 & n_2\end{matrix}\,\middle|\, z, \frac{k+2}{k+3}, x^{2(k+3)}\right],\\ W^+_k\left[\begin{matrix} n_4 & n_3\\ n_1 & n_2\end{matrix}\,\middle|\, z\right] &= z^{\frac{1}{2}\frac{k+3}{k+2}}\frac{g^+_k(z^{-1})}{g^+_k(z)}\hat{W}\left[\begin{matrix} n_4 & n_3\\ n_1 & n_2\end{matrix}\,\middle|\, z, \frac{k+3}{k+2}, x^{2(k+2)}\right],\end{aligned} \tag{2.4}$$

where[a]

$$
\begin{aligned}
\hat{W}\left[\begin{matrix} n\pm 2 & n\pm 1 \\ n\pm 1 & n \end{matrix}\middle| z,a,q\right] &= 1, \\
\hat{W}\left[\begin{matrix} n & n\pm 1 \\ n\pm 1 & n \end{matrix}\middle| z,a,q\right] &= z^{-a(1\pm n)}\frac{\Theta_q(q^a)\Theta_q(q^{\mp an}z)}{\Theta_q(q^{\mp an})\Theta_q(q^a z)}, \\
\hat{W}\left[\begin{matrix} n & n\mp 1 \\ n\pm 1 & n \end{matrix}\middle| z,a,q\right] &= q^{\mp a^2 n}z^{-a}\frac{\Theta_q(q^{a(1\mp n)})\Theta_q(z)}{\Theta_q(q^{\mp an})\Theta_q(q^a z)}, \\
\hat{W}\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\middle| z,a,q\right] &= 0 \text{ otherwise,}
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
f(z) &= z^{1/4}(x^3z;x^4)_\infty/(xz;x^4)_\infty, \\
g_k^+(z) &= \frac{(z;x^{2(k+2)},x^4)_\infty(x^{2(k+4)}z;x^{2(k+2)},x^4)_\infty}{(x^2z;x^{2(k+2)},x^4)_\infty(x^{2(k+3)}z;x^{2(k+2)},x^4)_\infty}, \\
g_k^-(z) &= \frac{(x^2z;x^{2(k+3)},x^4)_\infty(x^{2(k+4)}z;x^{2(k+3)},x^4)_\infty}{(x^4z;x^{2(k+3)},x^4)_\infty(x^{2(k+3)}z;x^{2(k+3)},x^4)_\infty}.
\end{aligned}
$$

The pairs (n_1,n_2), (n_2,n_3), (n_3,n_4), and (n_4,n_1) must satisfy RSOS admissibility conditions:

$$
\begin{aligned}
(m,n) \text{ is a } \kappa\text{-admissible pair} &\Leftrightarrow \begin{cases} 1\le m,n\le \kappa+1, \\ m=n\pm 1, \end{cases} \\
(m,n) \text{ is a } (\kappa,\lambda)\text{-admissible pair} &\Leftrightarrow \begin{cases} m+n=\lambda+2,\lambda+4,\dots,2\kappa-\lambda+2, \\ m-n=-\lambda,-\lambda+2,\dots,\lambda. \end{cases}
\end{aligned}
\tag{2.6}
$$

Adjacent lattice variables are $(k+1)$-admissible, and adjacent eigenstate labels are k-admissible.

Except usual relations as Yang–Baxter equation, crossing symmetry, and unitarity, the matrices $W_k^\pm$ satisfy the relation crucial in what follows:

$$
\sum_n W_{k+1}^+\left[\begin{matrix} n_4 & n \\ n_1 & n_2 \end{matrix}\middle| z\right]W_k^-\left[\begin{matrix} n_4 & n_3 \\ n & n_2 \end{matrix}\middle| z\right] = -\delta_{n_1n_3} \tag{2.7}
$$

if (n_1,n_2), (n_1,n_4), (n_2,n_3), and (n_3,n_4) pairs are $(k+1)$-admissible, and

$$
\begin{aligned}
r_{n_1}^\pm W_k^\pm\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\middle| z\right] &= r_{n_3}^\pm W_k^\pm\left[\begin{matrix} n_4 & n_1 \\ n_3 & n_2 \end{matrix}\middle| z\right], \\
r_n^\pm &= q_\mp^{n(a_\pm n-1)/2}\Theta_{q_\pm}(q_\mp^n),
\end{aligned}
\tag{2.8}
$$

where $a_\pm$, $q_\pm$ are defined for $W_k^\pm$ as it follows from Eq. (2.4):

$$
\begin{aligned}
a_+ &= (k+3)/(k+2), \qquad q_+ = x^{2(k+2)}. \\
a_- &= (k+2)/(k+3), \qquad q_- = x^{2(k+3)},
\end{aligned}
$$

Now let us define the fields $\phi_{p,p+1}(z)_{m\ n}^{m'n'}$ in the (q,x)-deformed case. Note that there is no natural way to define deformed fields ϕ_{pq}, contrary to the situation in the conformal field theory. We shall define the fields as follows[b]

$$
\begin{aligned}
\phi_{pq}(z)_{m\ n}^{m'n'} = \lim_{\substack{u_i\to z\\ v_j\to z}} \sum \mathcal{N}_p(u,v)\phi_{21}(x^{2-p}u_1)^{m'}\phi_{21}(x^{4-p}u_2)\dots\phi_{21}(x^{p-2}u_{p-1})_m \\
\times\phi_{12}(x^{q-2}v_1)^{n'}\phi_{12}(x^{q-4}v_2)\dots\phi_{12}(x^{2-q}v_p)_n,
\end{aligned}
\tag{2.9}
$$

---

[a] We are working in the gauge with negative $W\left[\begin{matrix} n & n\pm 1 \\ n\mp 1 & n \end{matrix}\middle| z\right]$ in contrast to Ref. 7.

[b] These vertex operators differ from those proposed recently by Kadeishvili.[15] In the trigonometric limit $k\to\infty$ the operators $\phi_{p,p+1}(z)$ give the restricted versions of Nakayashiki's vertex operators for the spin-inhomogeneous XXZ model.[16]

where the normalization factor is given by

$$\mathcal{N}_p(u,v)=\prod_{i<j}^{p-1}(g^+(x^{2(j-i)}u_j/u_i))^{-1}\prod_{i<j}^{q-1}(g^-(x^{2(i-j)}v_j/v_i))^{-1}\prod_i^{p-1}\prod_j^{q-1}f^{-1}(x^{p+q-2(i+j)}v_j/u_i). \tag{2.10}$$

The summation is taken over all intermediate states allowed by Eq. (2.1).

Note that all singular terms in this expression cancel out. Indeed, according to Ref. 7 the singularities appear in the pair products as follows

$$\begin{aligned}(g^+(x^2z/z'))^{-1}\phi_{21}(z')^m_{m\pm1}\phi_{21}(x^2z)^{m\pm1}_m&=\pm\frac{\text{const}}{1-z/z'}+O(1),\\ (g^-(x^{-2}z/z'))^{-1}\phi_{12}(x^2z')^n_{n\pm1}\phi_{12}(z)^{n\pm1}_n&=\pm\frac{\text{const}'}{1-z/z'}+O(1)\end{aligned}$$

as $z'\to z$. The summation over intermediate states $m\pm1$, $n\pm1$ cancels the singular terms.

In the commutation relations of the operators $\phi_{pq}(z)$

$$\phi_{p_1q_1}(z_1)^{m'n'}_{r\ \ s}\phi_{p_2q_2}(z_2)^{r\ \ s}_{mn}=\sum_{r's'}W^k_{p_1q_1p_2q_2}\left[\begin{matrix}m'n' & r's'\\ rs & mn\end{matrix}\,\middle|\,\frac{z_1}{z_2}\right]\phi_{p_2q_2}(z_2)^{m'n'}_{r'\ s'}\phi_{p_1q_1}(z_1)^{r'\ s'}_{mn} \tag{2.11}$$

the matrix $W^k_{p_1q_1p_2q_2}$ factors into three parts:

$$\begin{gathered}W^k_{p_1q_1p_2q_2}\left[\begin{matrix}m'n' & r's'\\ rs & mn\end{matrix}\,\middle|\,z\right]=W^+_{k,p_1-1,p_2-1}\left[\begin{matrix}m' & r'\\ r & m\end{matrix}\,\middle|\,z\right]W^-_{k,q_1-1,q_2-1}\left[\begin{matrix}n' & s'\\ s & n\end{matrix}\,\middle|\,z\right]\tau_{p_1q_1p_2q_2}(z),\\ W^\pm_{kll'}\left[\begin{matrix}n_4 & n_3\\ n_1 & n_2\end{matrix}\,\middle|\,z\right]=\sum_{\{\kappa_{ij},\ i,j\neq0\}}\prod_{i=1}^{l}\prod_{j=1}^{l'}W^\pm_k\left[\begin{matrix}\kappa_{ij} & \kappa_{i-1j}\\ \kappa_{ij-1} & \kappa_{i-1j-1}\end{matrix}\,\middle|\,x^{\pm2(i-j)}z\right],\\ \kappa_{0l}=n_1,\quad \kappa_{00}=n_2,\quad \kappa_{l0}=n_3,\quad \kappa_{ll}=n_4,\\ \tau_{p_1q_1p_2q_2}(z)=\prod_{i=1}^{p_1-1}\prod_{j=1}^{q_2-1}\tau(x^{p_1+q_2-2(i+j)}z)\prod_{i=1}^{p_2-1}\prod_{j=1}^{q_1-1}\tau(x^{p_2+q_1-2(i+j)}z)\end{gathered} \tag{2.12}$$

The matrices $W^\pm_{kll'}$ are the result of $l\times l'$ fusion[9] of the matrices $W^\pm_k$. The matrix $W^+_{kll'}$ is the weight matrix of the fusion RSOS model. In the case $p_1=p_2$, $q_1=q_2$ the matrices $W^\pm_{kl}$ of the fusion model with square faces appear:

$$W^\pm_{kl}\left[\begin{matrix}n_4 & n_3\\ n_1 & n_2\end{matrix}\,\middle|\,z\right]=W^\pm_{kll}\left[\begin{matrix}n_4 & n_3\\ n_1 & n_2\end{matrix}\,\middle|\,z\right]. \tag{2.13}$$

We need to make a remark. Usually the definition of the fusion W-matrices (2.12) implies that the summation is taken over all $\{\kappa_{ij}\}$ allowed by unrestricted SOS admissibility conditions.[1] We suppose that the summation is taken over $\{\kappa_{ij}\}$ satisfying *the RSOS admissibility conditions*. But it can be checked in simple cases (the first nontrivial one appears at $l=4$) that the configurations with internal indices that do not satisfy the RSOS admissibility conditions cancel each other. The mechanism of this cancellation is very similar to that of cancellation of inadmissible intermediate states in the bosonic representation of minimal conformal models.[17] In the general case we can argue in the following way. We know that the RSOS vertex operators are represented in the boson representation by the same operators as the SOS vertex operators do. Then the commutation relations of vertex operators must be given by the fusion weights in the usual sense. But if the in-state is admissible, i. e. belongs to a space $\mathcal{H}_{mn}$ with $m<k+1$, $n<k+2$, then the out-state and all internal states in the Eq. (2.9) are also admissible. Then commutation relation must contain W matrices with RSOS admissibility condition in all internal indices. Therefore, two types of W matrices coincide.

3. Fusion RSOS Models: Type I Vertex Operators

Recall the structure of the coset models (1.1). The vertex operators $\phi_{pp'q}(z)$ of the coset N_{kl} are labelled by three integers p, p', and q (see, for example, Ref. 18)

$$\begin{gathered}\phi^{(k,l)}_{pp'q}(z)^{m'\,\mu'\,n'}_{m\;\mu\;n}:\mathcal{H}_{m\mu n}\to\mathcal{H}_{m'\mu'n'},\\ p=1,\dots,k+1,\quad p'=1,\dots,l+1,\quad q=1,\dots,k+l+1,\\ p+p'-q-1\in 2\mathbf{Z}.\end{gathered}\tag{3.1}$$

Here $\mathcal{H}_{m\mu n}$ is a decomposable but not irreducible Virasoro algebra representation. The fusion rules are given by

$$\begin{gathered}|m-p|+1\le m'\le\min(m+p-1,k+3-m-p),\\ |\mu-p'|+1\le\mu'\le\min(\mu+p'-1,l+3-\mu-p'),\\ |n-q|+1\le n'\le\min(n+q-1,k+l+3-n-q).\end{gathered}\tag{3.2}$$

Consider the field $\phi_{1,l+1,l+1}(z)^{m'\,\mu'\,n'}_{m\;\mu\;n}$ (or, equivalently, $\phi_{k+1,1,k+1}$). We will omit the indices m, m', μ, and μ', because $m'=m$ and $\mu'=l+2-\mu$, and they produce no nontrivial braiding. This field has a simple representative in the coset construction of minimal models[12]

$$\phi_{1,l+1,l+1}\sim\phi^{(k)}_{12}\phi^{(k+1)}_{23}\dots\phi^{(k+l-1)}_{l,l+1}.$$

More precisely, let in the deformed case

$$\begin{aligned}\phi(z)^{n'}_{n}&\equiv\phi_{1,l+1,l+1}(z)^{n'}_{n}\\ &\sim\bigoplus_{\{m_i,m_i'\}}r^{(k+1)}_{m_1}r^{(k+2)}_{m_2}\dots r^{(k+l-1)}_{m_{l-1}}\phi^{(k)}_{12}(z)^{m_0'm_1'}_{m_0m_1}\phi^{(k+1)}_{23}(z)^{m_1'm_2'}_{m_1m_2}\dots\phi^{(k+l-1)}_{l,l+1}(z)^{m_{l-1}'n'}_{m_{l-1}n}\end{aligned}\tag{3.3}$$

(the terms with $m_0'=m_0$ only are nonvanishing), $r^{(\kappa)}_n=r^+_n$ for $q_+=x^{2(\kappa+2)}$. Using Eqs. (2.7) and (2.8) it is easy to check the following commutation relation

$$\phi(z_1)^{n'}_{s}\phi(z_2)^{s}_{n}=(-)^{l(l-1)/2}\sum_{s'}W^-_{K-1,l}\left[\begin{matrix}n'&s'\\ s&n\end{matrix}\middle|\frac{z_1}{z_2}\right]\phi(z_2)^{n'}_{s'}\phi(z_1)^{s'}_{n}.\tag{3.4}$$

The functions $\tau(z)$ drop out from this expression. The sign factor $(-1)^{l(l-1)/2}$ seems to be strange, but it has a clear physical meaning. Namely, it is easy to check that

$$W^-_{Kl}\left[\begin{matrix}n'&s'\\ s&n\end{matrix}\middle|1\right]=(-)^{l(l-1)/2}\delta_{ss'},$$

if the pairs (s,n), (n,s'), (s',n'), and (n',s) are (K,l)-admissible. It means that we need the factor $(-1)^{l(l-1)/2}$ to avoid the unphysical situation $\phi(z)^{n'}_{s}\phi(z)^{s}_{n}=0$. Note that this argument excludes the simplest field $\phi^{(k+l-1)}_{1,k+l+1}(z)$ from the candidates for the type I vertex operators.

4. Fusion RSOS Models: Type II Vertex Operators

Let us try to find the type II vertex operators. Two main properties of the type II vertex operators are: a) they commute with the type I vertex operators up to a function; b) they are 'elementary' fields, i. e. they generate in fusion all operators commuting with the type I vertex operators. We conjecture that they coincide with fields $\phi_{221}(z)^{m'\,\mu'\,n'}_{m\;\mu\;n}$. We will omit the indices n and n', because $n'=n$. More precisely, we write

$$\begin{gathered}\psi(z)^{m'\,\mu'}_{m\;\mu}\equiv\phi_{221}(z)^{m'\,\mu'}_{m\;\mu},\\ \bigoplus_{\{n_i,n_i'\}}r^{(2)}_{n_1}\dots r^{(l-1)}_{n_{l-2}}\xi(l,l-1)^{n_{l-1}'}_{n_{l-1}}r^{(l)}_{l+2-n_{l-1}}\phi^{(1)}_{12}(z)^{1n_1'}_{1n_1}\phi^{(2)}_{23}(z)\dots\phi^{(l-1)}_{l-1,l}(z)^{n_{l-2}'n_{l-1}'}_{n_{l-2}n_{l-1}}\\ \times\phi_{221}(z)^{l+2-n_{l-1}',\mu'}_{l+2-n_{l-1},\mu}\sim\phi^{(k)}_{21}(z)^{\mu'}_{\mu}.\end{gathered}\tag{4.1}$$

Here

$$\xi(\kappa,\lambda)_n^{n'} = \left[\frac{n+n'+\lambda}{2}\right]_{\lambda+1} \Big/ \left[\frac{n-n'+2\kappa-\lambda+2}{2}\right]_{\kappa+1-\lambda},$$

$$[u] = x^{(2u-\kappa-2)^2/4(\kappa+2)}\Theta_{x^{2(\kappa+2)}}(x^{2u}), \qquad [u]_t = \prod_{i=0}^{t-1}[u-i].$$

Eq. (4.1) defines the operator $\psi(x)$ uniquely.[12] Some comments are necessary. The subscript n_{l-1} in $\phi^{(l)}_{l-1,l}$ is paired with $l+2-n_{l-1}$ in ϕ_{221} in Eq. (4.1), not with another n_{l-1}. We need this to ensure the same commutation relation of both sides of Eq. (4.1) with the fermionic currents introduced below in Sec. 5. To construct this pairing we have to use the Bazhanov–Reshetikhin duality of the fusion W matrices.[19] It relates the matrices $W^-_{\kappa-1,\lambda,\lambda'}$ and $W^-_{\kappa-1,\lambda,\kappa-\lambda'}$. Applying this duality twice (once with respect to λ and once with respect to λ') and assuming $\lambda'=\lambda$ one can obtain

$$W^-_{\kappa-1,\lambda}\left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix}\middle| z\right] = (-)^{\kappa+(\kappa+1)(\kappa-2\lambda)/2}\frac{\xi(\kappa,\lambda)^{n_4}_{n_3}\xi(\kappa,\lambda)^{n_3}_{n_2}}{\xi(\kappa,\lambda)^{n_4}_{n_1}\xi(\kappa,\lambda)^{n_1}_{n_2}} W^-_{\kappa-1,\kappa-\lambda}\left[\begin{matrix} \kappa+2-n_1 & n_3 \\ n_1 & \kappa+2-n_2 \end{matrix}\middle| z\right].$$

This gives the following commutation relations:

$$\begin{aligned} \psi(z_1)_{r\ \rho}^{m'\mu'}\psi(z_2)_{m\mu}^{r\ \rho} &= -\sum_{r'\rho'} W_k^+\left[\begin{matrix} m' & r' \\ r & m \end{matrix}\middle| \frac{z_1}{z_2}\right] W_l^+\left[\begin{matrix} \mu' & \rho' \\ \rho & \mu \end{matrix}\middle| \frac{z_1}{z_2}\right]\psi(z_2)_{r'\ \rho'}^{m'\mu'}\psi(z_1)_{m\mu}^{r'\rho'}, \\ \phi(z_1)_n^{n'}\psi(z_2)_{m\ \mu}^{m'\mu'} &= \tau(z_1/z_2)\psi(z_2)_{m\ \mu}^{m'\mu'}\phi(z_1)_n^{n'}. \end{aligned} \tag{4.2}$$

¿From these commutation relations the following picture of excited states can be drawn. Any (N-particle) eigenvector of the transfer matrix $T_{kl}(z)$ is labelled by a chain of pairs of positive integers $\{(m_0,\mu_0),(m_1,\mu_1),\ldots,(m_{N-1},\mu_{N-1}),(m_N,\mu_N)\}$, $m_{i+1}=m_i\pm 1$, $\mu_{i+1}=\mu_i\pm 1$, $m_i\le k+1$, $\mu_i\le l+1$, and by a chain of complex numbers $\{z_1,\ldots,z_N\}$, $|z_i|=1$. The eigenvector is represented by the operator $\psi(z_N)^{m_N\ \ \ \mu_N}_{m_{N-1}\mu_{N-1}}\ldots\psi(z_1)^{m_1\mu_1}_{m_0\mu_0}$. The associated eigenvalue of the transfer matrix $T^{(k,l)}(z)$ is $\tau(z/z_N)\ldots\tau(z/z_1)$. The pairs (m_0,μ_0) and (m_N,μ_N) define conditions at the infinity. A pair (m,μ) corresponds to the configuration $\ldots,\, m+\mu-1,\, m+l+1-\mu,\, m+\mu-1,\, m+l+1-\mu,\,\ldots$ This coincides with the results of Ref. 10.

5. Notes on Traces of Vertex Operators

It is known that conformal blocks of coset models are given by solutions of some linear algebraic equations.[20–22] The traces of vertex operators satisfy similar equations. Here we make some notes concerning these equations.

Consider for example the zero-point correlation function of vertex operators $\chi^{(k,l)}_{pp'q} = \mathrm{Tr}_{\mathcal{H}_{pp'q}}(x^{4D_{k,l}})$ ($D_{k,l}$ is the grading operator with the same spectrum as the Virasoro generator L_0 in conformal field theory). In terms of these quantities the local state probabilities are expressed.[9] The local spin here is q, whereas p and p' are defined by the boundary conditions. It is evident that $\chi^{(k,l)}_{pp'q}$ satisfies the equation

$$\sum_{p'\{\mu\}}\chi^{(1)}_{\mu_0\mu_1}\ldots\chi^{(l-1)}_{\mu_{l-2},l+2-p'}\chi^{(k,l)}_{pp'q} = \sum_{\{m\}}\chi^{(k)}_{pm_1}\chi^{(k+1)}_{m_1m_2}\ldots\chi^{(k+l-1)}_{m_{l-1}q},$$

where $\chi^{(k)}_{pq} = \mathrm{Tr}_{\mathcal{H}_{pq}}(x^{4D_k})$ is the zero-point correlation function of an ABF model. Evidently, these equations do not define the amount $\chi^{(k,l)}_{pp'q}$ uniquely. We need more equations. To find them consider the field in the denominator of our coset construction

$$\begin{gathered} \sigma_s(z) = \frac{1}{N_s}\sum_{\lambda\lambda'} r^{(s)}_\lambda \phi^{(s)}_{12}(z)^{\lambda'}_\lambda \phi^{(s+1)}_{21}(z)^{\lambda'}_\lambda, \\ \sigma_s(z_1)\sigma_s(z_2) = -\sigma_s(z_2)\sigma_s(z_1), \qquad \sigma_{s\pm1}(z_1)\sigma_s(z_2) = \tau(z_1/z_2)\sigma_s(z_2)\sigma_{s\pm1}(z_1), \\ \sigma_{s'}(z_1)\sigma_s(z_2) = \sigma_s(z_2)\sigma_{s'}(z_1), \quad |s'-s|>1. \end{gathered} \tag{5.1}$$

Here N_s is a normalization factor. It can be chosen, for example, so that $\langle\sigma_s(1)\sigma_s(0)\rangle = 1$. The fields $\sigma_s(z)$ can be considered as additional currents. In conformal field theory it has the conformal dimension 1/2. In the conformal limit where the coset construction is studied rigorously it can be checked that it coincides with the field $\sigma_{k+s}(z)$ in the numerator

$$\sigma_s(z) = \sigma_{k+s}(z).$$

We conjecture that this identity holds in the deformed case. Let us introduce the trace

$$\Sigma^{(s)}_{n_1n_2n_3} = \mathrm{Tr}_{\mathcal{H}^{(s)}_{n_1n_2}\otimes\mathcal{H}^{(s+1)}_{n_2n_3}}\left(x^{4(D_s+D_{s+1})}\sigma_s(x^{-2}z)\sigma_s(z)\right),$$

which is independent of z. We obtain $l-1$ equations

$$\sum_{p'\{\mu\}}\chi^{(1)}_{\mu_0\mu_1}\ldots\Sigma^{(s)}_{\mu_{s-1}\mu_s\mu_{s+1}}\ldots\chi^{(l-1)}_{\mu_{l-2},l+2-p'}\chi^{(k,l)}_{pp'q} = \sum_{\{m\}}\chi^{(k)}_{pm_1}\chi^{(k+1)}_{m_1m_2}\ldots\Sigma^{(k+s)}_{m_{s-1}m_sm_{s+1}}\ldots\chi^{(k+l-1)}_{m_{l-1}q}. \tag{5.2}$$

We have l equations for $l+1$ variables $\chi^{(k,l)}_{pp'q}$. But $\chi^{(k,l)}_{pp'q} = 0$ unless $p+p'-q-1$ is odd. Hence, we have only $\lfloor(l+1)/2\rfloor$ or $\lfloor(l+2)/2\rfloor$ variables.

The similar procedure is necessary for other trace calculations. The origin of this feature is related to the fact that the set of the Virasoro charges does not form the complete chiral algebra of the product $M_1\ldots M_{l-1}$. We need additional currents. Fermionic fields $\sigma_s(z)$ play the role of these currents. But they cannot enter the character of this model in the usual way $\exp(i\theta(\text{charge}))$, and we need instead to consider additional traces $\Sigma^{(s)}_{n_1n_2n_3}$.

6. Conclusion

We considered the elliptic deformation of the coset construction (1.1) and gave evidences that it describes the fusion RSOS models in the framework of the Kyoto approach to integrable models of statistical mechanics. Despite absence of a rigorous proof it gives a plain recipe for calculation of correlation functions.

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